Appendix A Model Algebra, Proofs, and Calibration

A.1 Derivation of equation (2.2)

Type-(i,i') workers solve:

$$\max_{q_c,q_r,h} \quad B_i \left(\frac{Q_i}{\delta}\right)^{\delta}\left(\frac{h(i,i')}{1-\delta}\right)^{1-\delta} \qquad \text{s.t.} \quad w(i,i') = p_{ci}q_c(i,i') + p_{ri}q_r(i,i') + r_i h(i,i')$$

resulting in first order conditions (FOC):

$$\begin{aligned}
&[q_c(i,i')] \qquad \left(\frac{\delta}{1-\delta}\right)^{1-\delta} q_c(i,i')^{-\frac{1}{\sigma}} B_i Q_i^{\frac{1-\sigma(1-\delta)}{\sigma}} h(i,i')^{1-\delta} - \lambda p_{ci} = 0\\
&[q_r(i,i')] \qquad \left(\frac{\delta}{1-\delta}\right)^{1-\delta} q_r(i,i')^{-\frac{1}{\sigma}} B_i Q_i^{\frac{1-\sigma(1-\delta)}{\sigma}} h(i,i')^{1-\delta} - \lambda p_{ri} = 0\\
&[h(i,i')] \qquad \left(\frac{1-\delta}{\delta}\right)^{\delta} h^{-\delta} B_i Q_i^{\delta} - \lambda r_i = 0
\end{aligned}$$

where λ is the shadow price of the wage $w(i,i')$. Equating the FOCs for $q_c(i,i')$ and $q_r(i,i')$ then solving for $q_r(i,i')$ reveals:

$$q_r(i,i') = \left(\frac{p_{ri}}{p_{ci}}\right)^{-\sigma} q_c(i,i') \tag{A.1}$$

Equating the FOCs for $q_c(i,i')$ and $h(i,i')$ then solving for $h(i,i')$ implies:

$$h(i,i') = \left(\frac{1-\delta}{\delta}\right)\frac{p_{ci}^{-\sigma} q_c(i,i') P_i^{1-\sigma}}{r_i} \tag{A.2}$$

where $P_i \equiv (p_{ci}^{1-\sigma} + p_{ri}^{1-\sigma})^{\frac{1}{1-\sigma}}$ is the CES price index for location i.

Given the type-(i,i') worker's utility takes on a Cobb-Douglas form, we can express the total amount she spends on consumption goods as:

$$p_{ci}q_{ci} + p_{ri}q_{ri} = \delta w(i,i')$$

Substituting the above and (A.7) into the budget constraint and solving for $q_c(i,i')$ yields the worker's Marshallian demand for $q_c(i,i')$:

$$q_c(i,i') = \delta\left(\frac{p_{ci}}{P_i}\right)^{-\sigma}\frac{w(i,i')}{P_i} \tag{A.3}$$

Then, substituting (A.3) into (A.1) and (A.2) yields the worker's Marshallian demands for $q_r(i,i')$ and $h(i,i')$:

$$q_r(i,i') = \delta\left(\frac{p_{ri}}{P_i}\right)^{-\sigma}\frac{w(i,i')}{P_i} \tag{A.4}$$

$$h(i,i') = (1-\delta)\frac{w(i,i')}{r_i} \tag{A.5}$$

Finally, substituting equations (A.3) through (A.5) into the worker's utility function results in the indirect utility function equation (2.2):

$$V(i,i') = \frac{B_i w(i,i')}{P_i^{\delta} r_i^{1-\delta}}$$

A.2 Derivation of equation (2.6)

Following the set-up in the theoretical framework, letting $p_c \equiv 1$, and applying the stated simplifying assumptions that $\tau_{cr} = \tau_{rc} = \tau$, $L_r(c,r) = 0$, and $\kappa_{rc} = \kappa$ this model can be summarised by the following system of equations:

$$Y_c = A_c L_c^{1+\alpha} \tag{A.6}$$
$$Y_r = A_r L_r \tag{A.7}$$
$$w_c = A_c L_c^{\alpha} \tag{A.8}$$
$$w_r = p_r A_r \tag{A.9}$$
$$P_c = \left[1 + (\tau p_r)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \tag{A.10}$$
$$P_r = \left[\tau^{1-\sigma} + p_r^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \tag{A.11}$$
$$V(c,c) = \frac{B_c w_c}{P_c^{\delta} r_c^{1-\delta}} \tag{A.12}$$
$$V(r,c) = \frac{B_r w_c}{\kappa P_r^{\delta} r_r^{1-\delta}} \tag{A.13}$$
$$V(r,r) = \frac{B_r w_r}{P_r^{\delta} r_r^{1-\delta}} \tag{A.14}$$
$$h(c,c) = (1-\delta)\frac{w_c}{r_c} \tag{A.15}$$
$$h(r,c) = (1-\delta)\frac{w_c}{\kappa r_r} \tag{A.16}$$
$$h(r,r) = (1-\delta)\frac{w_r}{r_r} \tag{A.17}$$
$$X_c = \delta\left[\frac{1}{P_c^{1-\sigma}}\Big(w_c L_c(c,c)\Big) + \frac{\tau^{-\sigma}}{P_r^{1-\sigma}}\left(w_r L_r(r,r) + \frac{w_c}{\kappa} L_c(r,c)\right)\right] \tag{A.18}$$
$$X_r = \delta\left[\frac{(\tau p_r)^{-\sigma}}{P_c^{1-\sigma}}\Big(w_c L_c(c,c)\Big) + \frac{p_r^{-\sigma}}{P_r^{1-\sigma}}\left(w_r L_r(r,r) + \frac{w_c}{\kappa} L_c(r,c)\right)\right] \tag{A.19}$$

As per Definition 1, there are four equilibrium conditions:

1. Goods market clearing: $Y_c + Y_r = X_c + X_r$
2. Labour market clearing: $L_c + L_r = L_c(c,c) + L_c(r,c) + L_r(r,r) = \overline{L}$

3. Housing market clearing: $H_c = h(c,c)L_c(c,c)$ and $H_r = h(r,r)L_r(r,r) + h(r,c)L_c(r,c)$

4. No spatial arbitrage: $V(c,c) = V(r,c) = V(c,c) = \overline{V}$

By combining equilibrium requirements 1 through 4 with equations (A.6) through (A.19), equation (2.6) results.

Housing market clearing implies the total housing stock, which is exogenously determined, in a given location equals the total local demand for housing. Substituting equations (A.15) through (A.17) into equilibrium condition 3 reveal:

$$H_c = h(c,c)L_c(c,c) = \frac{(1-\delta)}{r_c} w_c L_c(c,c)$$

$$\implies r_c = \frac{(1-\delta)}{H_c} w_c L_c(c,c) \tag{A.20}$$

$$H_r = h(r,r)L_r(r,r) + h(r,c)L_c(r,c) = \frac{(1-\delta)}{r_c}\Big(w_r L_r(r,r) + \frac{w_c}{\kappa} L_c(r,c)\Big)$$

$$\implies r_r = \frac{(1-\delta)}{H_r}\Big(w_r L_r(r,r) + \frac{w_c}{\kappa} L_c(r,c)\Big) \tag{A.21}$$

Substituting equation (A.20) into equation (A.12) and equation (A.21) into equations (A.13) and (A.14) yield:

$$V(c,c) = \frac{B_c H_c^{1-\delta} w_c^{\delta}}{(1-\delta)^{1-\delta} P_c^{\delta} L_c(c,c)^{1-\delta}} \tag{A.22}$$

$$V(r,c) = \frac{B_r H_r^{1-\delta} w_c}{\kappa (1-\delta)^{1-\delta} P_r^{\delta}\Big(w_r L_r(r,r) + \frac{w_c}{\kappa} L_c(r,c)\Big)^{1-\delta}} \tag{A.23}$$

$$V(r,r) = \frac{B_r H_r^{1-\delta} w_r}{(1-\delta)^{1-\delta} P_r^{\delta}\Big(w_r L_r(r,r) + \frac{w_c}{\kappa} L_c(r,c)\Big)^{1-\delta}} \tag{A.24}$$

No spatial arbitrage (equilibrium condition 4) implies that $V(c,c) = \overline{V}$, $V(r,r) = V(r,c)$, and $V(r,r) = V(c,c)$ hold simultaneously. The first equality between equation (A.22) and $\overline{V}$ implies:

$$V(c,c) = \overline{V} : \frac{B_c H_c^{1-\delta} w_c^{\delta}}{(1-\delta)^{1-\delta} P_c^{\delta} L_c(c,c)^{1-\delta}} = \overline{V}$$

$$\implies L_c(c,c) = \frac{B_c^{\frac{1}{1-\delta}} H_c w_c^{\frac{\delta}{1-\delta}}}{(1-\delta) P_c^{\frac{\delta}{1-\delta}} \overline{V}^{\frac{1}{1-\delta}}} \tag{A.25}$$

The second equality between equations (A.23) and (A.24) reveals:

$$V(r,r) = V(r,c): \frac{B_r H_r^{1-\delta} w_r}{(1-\delta)^{1-\delta} P_r^{\delta}\Big(w_r L_r(r,r) + \frac{w_c}{\kappa} L_c(r,c)\Big)^{1-\delta}}$$
$$= \frac{B_r H_r^{1-\delta} w_c}{\kappa(1-\delta)^{1-\delta} P_r^{\delta}\Big(w_r L_r(r,r) + \frac{w_c}{\kappa} L_c(r,c)\Big)^{1-\delta}}$$
$$\implies w_r = \frac{w_c}{\kappa} \tag{A.26}$$

After substituting equation (A.26) into equation (A.24), the third equality between (A.22) into equation (A.24) implies:

$$V(r,r) = V(c,c): \frac{B_r H_r^{1-\delta} w_c^{\delta}}{\kappa^{\delta}(1-\delta)^{1-\delta} P_r^{\delta}\Big(L_r(r,r) + L_c(r,c)\Big)^{1-\delta}}$$
$$= \frac{B_c H_c^{1-\delta} w_c^{\delta}}{(1-\delta)^{1-\delta} P_c^{\delta} L_c(c,c)^{1-\delta}}$$
$$\implies L_r(r,r) + L_c(r,c) = \frac{B_r^{\frac{1}{1-\delta}} H_r P_c^{\frac{\delta}{1-\delta}}}{\kappa^{\frac{\delta}{1-\delta}} B_c^{\frac{1}{1-\delta}} H_c P_r^{\frac{\delta}{1-\delta}}} L_c(c,c)$$

and given the result in equation (A.25), it follows the above becomes:

$$\implies L_r(r,r) + L_c(r,c) = \frac{B_r^{\frac{1}{1-\delta}} H_r w_c^{\frac{\delta}{1-\delta}}}{(1-\delta)\kappa^{\frac{\delta}{1-\delta}} P_r^{\frac{\delta}{1-\delta}} \overline{V}^{\frac{1}{1-\delta}}} \tag{A.27}$$

Adding goods demand equations (A.18) and (A.19) then substituting in results from equations (A.25), (A.26), and (A.27) as well as the price index equations (A.10) and (A.11), total goods demand is expressed as:

$$X_c + X_r = \delta\left[\left(\frac{1+(\tau p_r)^{-\sigma}}{P_c^{1-\sigma}}\right) w_c L_c(c,c) + \left(\frac{\tau^{-\sigma}+p_r^{-\sigma}}{P_r^{1-\sigma}}\right)\left(w_r L_r(r,r) + \frac{w_c}{\kappa} L_c(r,c)\right)\right]$$
$$= \delta\left[\left(\frac{1+(\tau p_r)^{-\sigma}}{P_c^{1-\sigma}}\right) w_c L_c(c,c) + \left(\frac{\tau^{-\sigma}+p_r^{-\sigma}}{P_r^{1-\sigma}}\right)\frac{w_c}{\kappa}\Big(L_r(r,r) + L_c(r,c)\Big)\right]$$
$$= \left(\frac{\delta}{1-\delta}\right)\left(\frac{w_c}{\overline{V}}\right)^{\frac{1}{1-\delta}}\left[\left(\frac{1+(\tau p_r)^{-\sigma}}{P_c^{1-\sigma}}\right)\frac{B_c^{\frac{1}{1-\delta}} H_c}{P_c^{\frac{\delta}{1-\delta}}} + \left(\frac{\tau^{-\sigma}+p_r^{-\sigma}}{P_r^{1-\sigma}}\right)\frac{B_r^{\frac{1}{1-\delta}} H_r}{\kappa^{\frac{1}{1-\delta}} P_r^{\frac{\delta}{1-\delta}}}\right]$$
$$= \left(\frac{\delta}{1-\delta}\right)\left(\frac{w_c}{\overline{V}}\right)^{\frac{1}{1-\delta}}\left[\frac{(1+(\tau p_r)^{-\sigma})B_c^{\frac{1}{1-\delta}} H_c}{(1+(\tau p_r)^{1-\sigma})^{\frac{\sigma(1-\delta)-1}{(\sigma-1)(1-\delta)}}} + \frac{(\tau^{-\sigma}+p_r^{-\sigma})B_r^{\frac{1}{1-\delta}} H_r}{\kappa^{\frac{1}{1-\delta}}(\tau^{1-\sigma}+p_r^{1-\sigma})^{\frac{\sigma(1-\delta)-1}{(\sigma-1)(1-\delta)}}}\right] \tag{A.28}$$

The equilibrium urban wage, equation (A.10), can be rearranged to reveal:

$$L_c = L_c(c,c) + L_c(r,c) = \left(\frac{w_c}{A_c}\right)^{\frac{1}{\alpha}}$$
$$\implies L_c(r,c) = \left(\frac{w_c}{A_c}\right)^{\frac{1}{\alpha}} - L_c(c,c) \tag{A.29}$$

Substituting this result into the labour supply equilibrium condition, condition 2, and rearranging reveals:

$$L_r(r,r) = \overline{L} - \left(\frac{w_c}{A_c}\right)^{\frac{1}{\alpha}}$$
$$\implies L_r = \overline{L} - \left(\frac{w_c}{A_c}\right)^{\frac{1}{\alpha}} \tag{A.30}$$

Given the above, summing equations (A.6) and (A.7) and substituting equations (A.29) and (A.30) into the result, total goods supply can be expressed:

$$Y_c + Y_r = A_c L_c^{1+\alpha} + A_r L_r$$
$$= A_r\overline{L} + \frac{w_c^{\frac{1+\alpha}{\alpha}}}{A_c^{\frac{1}{\alpha}}} - \frac{A_r w_c^{\frac{1}{\alpha}}}{A_c^{\frac{1}{\alpha}}} \tag{A.31}$$

Equating the total goods supply equation (A.31) and total goods demand equation (A.28) (i.e. the goods market clearing, condition 1) yields:

$$A_r\overline{L} + \frac{w_c^{\frac{1+\alpha}{\alpha}}}{A_c^{\frac{1}{\alpha}}} - \frac{A_r w_c^{\frac{1}{\alpha}}}{A_c^{\frac{1}{\alpha}}} =$$
$$\left(\frac{\delta}{1-\delta}\right)\left(\frac{w_c}{\overline{V}}\right)^{\frac{1}{1-\delta}}\left[\frac{(1+(\tau p_r)^{-\sigma})B_c^{\frac{1}{1-\delta}}H_c}{(1+(\tau p_r)^{1-\sigma})^{\frac{\sigma(1-\delta)-1}{(\sigma-1)(1-\delta)}}} + \frac{(\tau^{-\sigma}+p_r^{-\sigma})B_r^{\frac{1}{1-\delta}}H_r}{\kappa^{\frac{1}{1-\delta}}(\tau^{1-\sigma}+p_r^{1-\sigma})^{\frac{\sigma(1-\delta)-1}{(\sigma-1)(1-\delta)}}}\right]$$

and since equations (A.9) and (A.26) imply $p_r = \frac{w_r}{A_r} = \frac{w_c}{\kappa A_r}$, it follows the above is:

$$A_r\overline{L} + \frac{w_c^{\frac{1+\alpha}{\alpha}}}{A_c^{\frac{1}{\alpha}}} - \frac{A_r w_c^{\frac{1}{\alpha}}}{A_c^{\frac{1}{\alpha}}} =$$
$$\left(\frac{\delta}{1-\delta}\right)\left(\frac{w_c}{\overline{V}}\right)^{\frac{1}{1-\delta}}\left[\frac{(1+(\frac{\tau w_c}{\kappa A_r})^{-\sigma})B_c^{\frac{1}{1-\delta}}H_c}{(1+(\frac{\tau w_c}{\kappa A_r})^{1-\sigma})^{\frac{\sigma(1-\delta)-1}{(\sigma-1)(1-\delta)}}} + \frac{(\tau^{-\sigma}+(\frac{w_c}{\kappa A_r})^{-\sigma})B_r^{\frac{1}{1-\delta}}H_r}{\kappa^{\frac{1}{1-\delta}}(\tau^{1-\sigma}+(\frac{w_c}{\kappa A_r})^{1-\sigma})^{\frac{\sigma(1-\delta)-1}{(\sigma-1)(1-\delta)}}}\right]$$

Multiplying both sides by $(\frac{1-\delta}{\delta})\overline{V}^{\frac{1}{1-\delta}}w_c^{-\sigma}$ and some algebra reveal the equilibrium condition equation (2.6):

$$\overline{V}^{\frac{1}{1-\delta}}\left(\frac{1-\delta}{\delta}\right)\left[\frac{A_r\overline{L}}{w_c^{\sigma}}+\frac{w_c^{\frac{1}{\alpha}-\sigma}}{A_c^{\frac{1}{\alpha}}}(w_c-A_r)\right]=$$
$$\frac{B_c^{\frac{1}{1-\delta}}H_c(1+(\frac{\tau w_c}{\kappa A_r})^{-\sigma})}{\left(w_c^{\sigma-1}+(\frac{\kappa A_r}{\tau})^{\sigma-1}\right)^{\frac{\sigma(1-\delta)-1}{(\sigma-1)(1-\delta)}}}+\frac{B_r^{\frac{1}{1-\delta}}H_r(\tau^{-\sigma}+(\frac{w_c}{\kappa A_r})^{-\sigma})}{\kappa^{\frac{1}{1-\delta}}\left((\frac{w_c}{\tau})^{\sigma-1}+(\kappa A_r)^{\sigma-1}\right)^{\frac{\sigma(1-\delta)-1}{(\sigma-1)(1-\delta)}}}$$

A.3 Proof of Lemma 1

Proof of (i)

Proof. Consider equation (2.6). Subtract the right-hand-side (RHS) from the left-hand-side (LHS).

$$\underbrace{\left[\left(\frac{1-\delta}{\delta}\right)A_r\overline{L}\,\overline{V}^{\frac{1}{1-\delta}}-\frac{B_c^{\frac{1}{1-\delta}}H_c(\frac{\kappa A_r}{\tau})^{\sigma}}{\left(w_c^{\sigma-1}+(\frac{\kappa A_r}{\tau})^{\sigma-1}\right)^{\frac{\sigma(1-\delta)-1}{(\sigma-1)(1-\delta)}}}-\frac{B_r^{\frac{1}{1-\delta}}H_r(\kappa A_r)^{\sigma}}{\kappa^{\frac{1}{1-\delta}}\left((\frac{w_c}{\tau})^{\sigma-1}+(\kappa A_r)^{\sigma-1}\right)^{\frac{\sigma(1-\delta)-1}{(\sigma-1)(1-\delta)}}}\right]w_c^{-\sigma}}_{①}$$

$$+\underbrace{\left(\frac{1-\delta}{\delta}\right)\frac{\overline{V}^{\frac{1}{1-\delta}}}{A_c^{\frac{1}{\alpha}}}w_c^{\frac{1}{\alpha}-\sigma}(w_c-A_r)}_{②}$$

$$+\underbrace{\left[-\frac{B_c^{\frac{1}{1-\delta}}H_c}{\left(w_c^{\sigma-1}+(\frac{\kappa A_r}{\tau})^{\sigma-1}\right)^{\frac{\sigma(1-\delta)-1}{(\sigma-1)(1-\delta)}}}-\frac{B_r^{\frac{1}{1-\delta}}H_r}{\tau^{\sigma}\kappa^{\frac{1}{1-\delta}}\left((\frac{w_c}{\tau})^{\sigma-1}+(\kappa A_r)^{\sigma-1}\right)^{\frac{\sigma(1-\delta)-1}{(\sigma-1)(1-\delta)}}}\right]}_{③}=0$$

Define the resulting LHS (i.e., the sum of components 1 through 3) as function $f:\mathbb{R}_{++}\to\mathbb{R}$, which is nonlinear in a single unknown $w_c\in\mathbb{R}_{++}$. We are interested in values of w_c, labelled w_c^*, at which $f(w_c^*)=0$ thereby implying equation (2.6) holds.

By inspection, f is continuous for all $w_c\in\mathbb{R}_{++}$. Define f_i as the component of f labelled above as $i\in\{1,2,3\}$, let the lower bound on H_r in equation (2.7) be summarised as:

$$\Psi\equiv\left(\frac{B_c}{B_r}\frac{\kappa}{\tau}\right)^{\frac{1}{1-\delta}}\left[\left(\frac{1-\delta}{\delta}\right)\left(\frac{\overline{L}^{1-\delta}}{A_r^{\delta}}\frac{\overline{V}}{B_c}\frac{\tau}{\kappa}\right)^{\frac{1}{1-\delta}}-H_c\right]$$

and set $\zeta\equiv\left(\frac{1-\delta}{\delta}\right)\frac{\overline{V}^{\frac{1}{1-\delta}}}{A_c^{\frac{1}{\alpha}}}$, which is strictly positive given its components are all assumed positive. Evaluating the limit of each component of f as w_c approaches zero from the right, observe:

$$\lim_{w_c \to 0^+} f_1 = \underbrace{\Big(B_r A_r\Big)^{\frac{1}{1-\delta}}}_{>0} \underbrace{\Big(\Psi - H_r\Big)}_{<0 \text{ since } \Psi < H_r} * \infty = -\infty$$

$$\lim_{w_c \to 0^+} f_2 = \begin{cases} -\zeta * A_r * 0 = 0 & \text{if } \sigma < \frac{1}{\alpha} \\ -\zeta * A_r < 0 & \text{if } \sigma = \frac{1}{\alpha} \\ -\zeta * A_r * \infty = -\infty & \text{if } \frac{1}{\alpha} < \sigma < \frac{1+\alpha}{\alpha} \end{cases}$$

$$\lim_{w_c \to 0^+} f_3 = -\frac{H_c B_c^{\frac{1}{1-\delta}} \tau^{\frac{\sigma(1-\delta)-1}{1-\delta}}}{(\kappa A_r)^{\frac{\sigma(1-\delta)-1}{1-\delta}}} - \frac{H_r B_r^{\frac{1}{1-\delta}}}{(\tau\kappa)^{\sigma} A_r^{\frac{\sigma(1-\delta)-1}{1-\delta}}} < 0$$

Since f is linear in f_1, f_2, and f_3, it follows that $\lim_{w_c \to 0^+} f = \lim_{w_c \to 0^+} f_1 + \lim_{w_c \to 0^+} f_2 + \lim_{w_c \to 0^+} f_3$. Thus:

$$\lim_{w_c \to 0^+} f = -\infty$$

Then, evaluating the limit of each component as w_c approaches positive infinity, we find:

$$\lim_{w_c \to \infty} f_1 = \Big(B_r A_r\Big)^{\frac{1}{1-\delta}} \Big(\Psi - H_r\Big) * 0 = 0$$

$$\lim_{w_c \to \infty} f_2 = \begin{cases} \zeta * \lim_{w_c \to \infty} w_c^{\frac{1}{\alpha} - \sigma} * \lim_{w_c \to \infty} (w_c - A_r) = \infty & \text{if } \sigma < \frac{1}{\alpha} \\ \zeta * \lim_{w_c \to \infty} (w_c - A_r) = \infty & \text{if } \sigma = \frac{1}{\alpha} \\ \zeta * \lim_{w_c \to \infty} \frac{w_c - A_r}{w_c^{\sigma - \frac{1}{\alpha}}} \overset{\widehat{\text{H}}}{=} \mu * \lim_{w_c \to \infty} \frac{w_c^{\frac{1+\alpha}{\alpha} - \sigma}}{\sigma - \frac{1}{\alpha}} = \infty & \text{if } \frac{1}{\alpha} < \sigma < \frac{1+\alpha}{\alpha} \end{cases}$$

$$\lim_{w_c \to \infty} f_3 = \begin{cases} -H_c {B_c}^{\sigma} - H_r \left(\frac{B_r}{\tau\kappa}\right)^{\sigma} < 0 & \text{if } \frac{1}{1-\delta} = \sigma \\ 0 & \text{if } \frac{1}{1-\delta} < \sigma \end{cases}$$

where $\widehat{\text{H}}$ denotes application of L'Hôpital's rule. Again, linearity of f in f_1, f_2, and f_3 implies

$\lim_{w_c\to\infty} f = \lim_{w_c\to\infty} f_1 + \lim_{w_c\to\infty} f_2 + \lim_{w_c\to\infty} f_3$, so it follows that:

$$\lim_{w_c\to\infty} f = \infty$$

Given the continuity and limit behaviour of f, Bolzano's intermediate value theorem applies and there must exist at least one $w_c^* \in \mathbb{R}_{++}$ that satisfies equation (2.6). ■

Proof of (ii)

Proof. Substituting $\frac{1}{1-\delta}$ for σ, equation (2.6) reduces to:

$$\overline{V}^{\frac{1}{1-\delta}}\left(\frac{1-\delta}{\delta}\right)\left[\frac{A_r\overline{L}}{w_c^{\frac{1}{1-\delta}}} + \frac{w_c^{\frac{1}{\alpha}-\frac{1}{1-\delta}}}{A_c^{\frac{1}{\alpha}}}(w_c - A_r)\right] =$$
$$B_c{}^{\frac{1}{1-\delta}}H_c\left(1+\left(\frac{\tau w_c}{\kappa A_r}\right)^{-\frac{1}{1-\delta}}\right) + \frac{B_r{}^{\frac{1}{1-\delta}}H_r(\tau^{-\frac{1}{1-\delta}} + (\frac{w_c}{\kappa A_r})^{-\frac{1}{1-\delta}})}{\kappa^{\frac{1}{1-\delta}}}$$

Subtracting the RHS from the LHS yields:

$$\underbrace{\left(B_rA_r\right)^{\frac{1}{1-\delta}}\left(\Psi - H_r\right)w_c^{-\frac{1}{1-\delta}}}_{①} + \underbrace{\left(\frac{1-\delta}{\delta}\right)\frac{\overline{V}^{\frac{1}{1-\delta}}}{A_c^{\frac{1}{\alpha}}}w_c^{\frac{1}{\alpha}-\frac{1}{1-\delta}}(w_c - A_r)}_{②}$$
$$+\underbrace{\left(-H_cB_c{}^{\frac{1}{1-\delta}} - H_r\left(\frac{B_r}{\tau\kappa}\right)^{\frac{1}{1-\delta}}\right)}_{③} = 0$$

Define the resulting LHS as a function $\widetilde{f}:\mathbb{R}_{++}\to\mathbb{R}$. Note that $\widetilde{f}$ is a special case of f from the proof of **(i)** above. Therefore, it follows that $\widetilde{f}$ is continuous over its entire domain and:

$$\lim_{w_c\to 0^+} \widetilde{f} = -\infty$$
$$\lim_{w_c\to\infty} \widetilde{f} = \infty$$

so Bolzano's intermediate value theorem holds and there exist one or more $w_c = w_c^*$, such that $\widetilde{f}(w_c) = 0$.

Let $\widetilde{f}_i$ be the component of $\widetilde{f}$ labelled above as $i \in \{1, 2, 3\}$, with Ψ and ζ defined the same as in the proof of **(i)**. Taking the first derivative of each component with respect to w_c

reveals:

$$\frac{\mathrm{d}\widetilde{f}_1}{\mathrm{d}w_c} = \underbrace{-\left(\frac{1}{1-\delta}\right)}_{<0}\underbrace{(B_r A_r)^{\frac{1}{1-\delta}}}_{>0}\underbrace{(\Psi - H_r)}_{<0} w_c^{-\left(\frac{2-\delta}{1-\delta}\right)} > 0 \;\forall\; w_c \in \mathbb{R}_{++}$$

$$\frac{\mathrm{d}\widetilde{f}_2}{\mathrm{d}w_c} = \zeta\left[\underbrace{\left(\frac{1+\alpha}{\alpha} - \frac{1}{1-\delta}\right)}_{>0} w_c^{\gamma_1} + \underbrace{\left(\frac{1}{1-\delta} - \frac{1}{\alpha}\right)}_{>0} w_c^{\gamma_2}\right] > 0 \;\forall\; w_c \in \mathbb{R}_{++}$$

$$\frac{\mathrm{d}\widetilde{f}_3}{\mathrm{d}w_c} = 0 \;\forall\; w_c \in \mathbb{R}_{++}$$

where $\gamma_1 \equiv \frac{(1+\alpha)(1-\delta)-\alpha(2-\delta)}{\alpha(1-\delta)}$ and $\gamma_2 \equiv \frac{(1-\delta)-\alpha(2-\delta)}{\alpha(1-\delta)}$. Since $\widetilde{f}$ is linear in its components $\widetilde{f}_1$, $\widetilde{f}_2$, and $\widetilde{f}_3$, it follows that $\frac{\mathrm{d}\widetilde{f}}{\mathrm{d}w_c} > 0 \;\forall\; w_c \in \mathbb{R}_{++}$. As such, $\widetilde{f}$ is strictly monotone increasing and so $\widetilde{f}$ can equal nought no more than once, which in turn implies the value of $w_c = w_c^*$ where $\widetilde{f}(w_c^*) = 0$ must be unique. ■

In Figure A.1, I sketch generic forms the function f can take under Lemma 1(i) and (ii). Figure A.1a shows that although solutions can exist, under (i) none of these solutions are guaranteed to be unique. Under 1(ii), a generic form of which is sketched in Figure A.1b, a solution is guaranteed to exist and will be unique.

A.4 Proof of Lemma 2

Proof. As I show in Online Appendix A.2, $L_c(c,c)$, $L_c(r,c)$, and $L_r(r,r)$ can be expressed as functions of w_c in equilibrium (equations (A.25), (A.29), and (A.30), respectively). Given equation (A.25) implies $L_c(c,c)$ is positive for all positive values of the urban wage, for the spatial equilibrium to be regular, it must be that $L_c(c,c) < \overline{L}$, with the difference $\overline{L} - L_c(c,c)$ divided between commuters and rural worker types, i.e. $L_c(r,c) > 0$ and $L_r(r,r) > 0$.

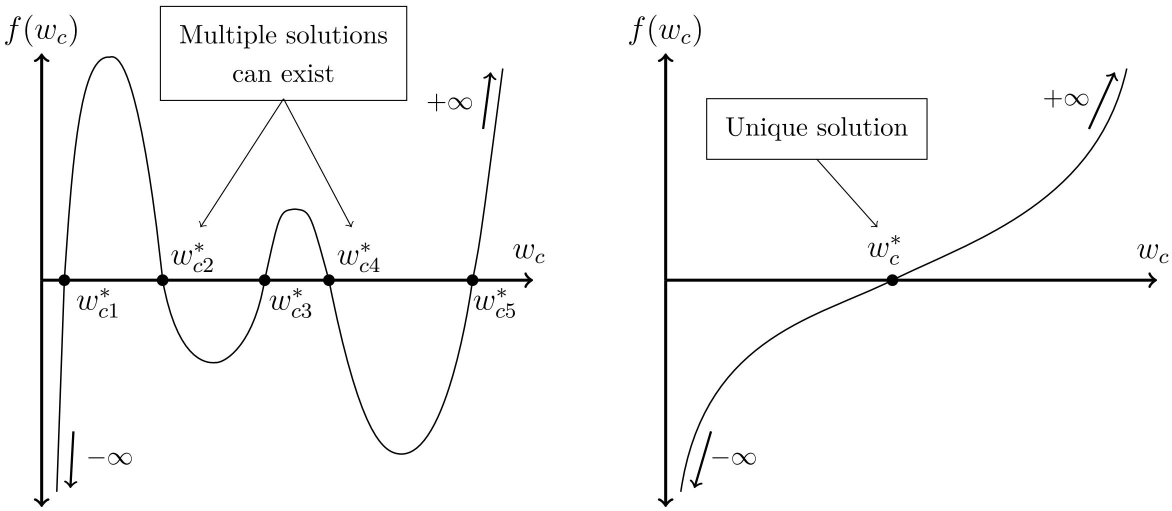
Conditions for $\boldsymbol{L_c(c,c) < \overline{L}}$. Rewriting equation (A.25) yields:

$$\begin{aligned} L_c(c,c) &= \frac{B_c^{\frac{1}{1-\delta}} H_c w_c^{\frac{\delta}{1-\delta}}}{(1-\delta)\left(1 + (\frac{w_c\tau}{\kappa A_r})^{1-\sigma}\right)^{\frac{\delta}{1-\delta}} \overline{V}^{\frac{1}{1-\delta}}} \\ &= \left(w_c^{\sigma-1} + \left(\frac{\kappa A_r}{\tau}\right)^{\sigma-1}\right)^{\frac{\delta}{(\sigma-1)(1-\delta)}} \left(\frac{H_c}{1-\delta}\right)\left(\frac{B_c}{\overline{V}}\right)^{\frac{1}{1-\delta}} \end{aligned}$$

Imposing $L_c(c,c) < \overline{L}$ and solving for w_c reveals an upper bound on w_c to ensure that the entire region's population does not live and work in the city:

$$w_c < \left(\Omega^{\sigma-1} - \left(\frac{\kappa A_r}{\tau}\right)^{\sigma-1}\right)^{\frac{1}{\sigma-1}} \tag{A.32}$$

Figure A.1: Existence and Uniqueness of Solutions under Lemma 1



(a) Existence of solution(s) under Lemma 1(i) **(b)** Unique solution under Lemma 1(ii)

Notes: These figures sketch generic forms the function f (i.e., the LHS of Equation 2.6 subtracted from the RHS of 2.6) can take under the restrictions in Lemma 1(i) and (ii). A solution to equation (2.6) is a value of w_c^* such that $f(w_c) = 0$. The w_c-axis represents the strictly positive wages $w_c \in \mathbb{R}_{++}$ earned by workers in the city, while the y-axis represents the values f can take on when evaluated at w_c, $f(w_c) \in \mathbb{R}$. The arrows in the bottom left and top right indicate the behaviour of f as w_c approaches zero from the right and positive infinity, respectively. Under Lemma 1(i), solutions to equation (2.6) can exist, but are not guaranteed to exist, while under (ii) a solution exists and is unique.

where $\Omega \equiv \left((1-\delta)\ \overline{L}/H_c\right)^{\frac{1-\delta}{\delta}} \left(\overline{V}/B_c\right)^{\frac{1}{\delta}}$. Since we restrict our attention only to positive values of w_c^*, it must be the case that $\left(\Omega^{\sigma-1} - (\kappa A_r/\tau)^{\sigma-1}\right)^{\frac{1}{\sigma-1}} > 0$, which is true only if the total region's population is sufficiently large:

$$\overline{L} > \left(\frac{H_c}{1-\delta}\right) \left(\frac{\kappa A_r}{\tau}\right)^{\frac{\delta}{1-\delta}} \left(\frac{B_c}{\overline{V}}\right)^{\frac{1}{1-\delta}}$$

This is condition (2.8) in in Lemma 2.

Conditions for $L_r(r, r) > 0$. Imposing the restriction that equation (A.30) must be greater than zero reveals:

$$w_c < A_c \overline{L}^{\alpha} \tag{A.33}$$

Then, upper bound on w_c for both $L_c(c, c)$ and $L_r(r, r)$ to be positive in equilibrium depends on the size of A_c. There are two cases:

(i) **Case #1:** If

$$A_c < \overline{L}^{-\alpha} \left(\Omega^{\sigma-1} - \left(\frac{\kappa A_r}{\tau} \right)^{\sigma-1} \right)^{\frac{1}{\sigma-1}}$$

then w_c is bounded above by $A_c \overline{L}^{\alpha}$

(ii) **Case #2:** If

$$A_c \geq \overline{L}^{-\alpha} \left(\Omega^{\sigma-1} - \left(\frac{\kappa A_r}{\tau} \right)^{\sigma-1} \right)^{\frac{1}{\sigma-1}}$$

then w_c is bounded above by $\left(\Omega^{\sigma-1} - \left(\frac{\kappa A_r}{\tau} \right)^{\sigma-1} \right)^{\frac{1}{\sigma-1}}$

Conditions for $\boldsymbol{L_c(r,c) > 0}$. Substituting equation A.25 into equation A.29 reveals:

$$L_c(r,c) = \left(\frac{w_c}{A_c} \right)^{\frac{1}{\alpha}} - \left(w_c^{\sigma-1} + \left(\frac{\kappa A_r}{\tau} \right)^{\sigma-1} \right)^{\frac{\delta}{(\sigma-1)(1-\delta)}} \left(\frac{H_c}{1-\delta} \right) \left(\frac{B_c}{\overline{V}} \right)^{\frac{1}{1-\delta}} \tag{A.34}$$

Equation (A.34) is continuous for all $w_c^* \in \mathbb{R}_{++}$, but unlike the equilibrium equation for $L_r(r,r)$, equation (A.34) is nonlinear in w_c, so there is not (necessarily) a unique set of bounds on w_c^* which ensures $L_c(r,c) > 0$. However, we require only that such a set of w_c exists where $L_c(r,c)(w_c) > 0$ that falls within the bounds of where $L_c(c,c) > 0$ and $L_r(r,r) > 0$ to identify the values of w_c for which a regular spatial equilibrium can exist. As such, we can study the behaviour of equation (A.34) at the bounds on w_c discussed above and again apply Bolzano's intermediate value theorem to show at least one set can exist under certain assumptions.

Given we are assuming that the equilibrium wage can only be strictly positive (i.e., $w_c^* > 0$), we first analyse the limit of $L_c(r,c)$ as it approaches nought from the right:

$$\lim_{w_c \to 0^+} L_c(r,c) = -\left(\frac{\kappa A_r}{\tau} \right)^{\frac{\delta}{1-\delta}} \left(\frac{H_c}{1-\delta} \right) \left(\frac{B_c}{\overline{V}} \right)^{\frac{1}{1-\delta}} < 0$$

Since the lower limit is less than zero, if the upper limit is greater than zero, Bolzano's intermediate value theorem tells us there exists at least one point w_c at which $L_c(r,c)(w_c) = 0$, implying the set of values greater than this point and less than the upper bound is a set where $L_c(r,c) > 0$. Recall there are two upper bounds on w_c under which both $L_c(c,c) > 0$ and $L_r(r,r) > 0$ depending on the relative size of A_c described above. Thus, we must evaluate both cases to check the conditions under which $L_c(r,c) > 0$ as well.

(i) **Case #1:** $A_c \overline{L}^{\alpha}$ is the upper bound and evaluating the limit of $L_c(r,c)$ as w_c approaches $A_c \overline{L}^{\alpha}$ reveals:

$$\lim_{w_c \to A_c \overline{L}^{\alpha}} L_c(r,c) = \overline{L} - \left((A_c \overline{L}^{\alpha})^{\sigma-1} + \left(\frac{\kappa A_r}{\tau} \right)^{\sigma-1} \right)^{\frac{\delta}{(\sigma-1)(1-\delta)}} \left(\frac{H_c}{1-\delta} \right) \left(\frac{B_c}{\overline{V}} \right)^{\frac{1}{1-\delta}}$$

This limit is greater than zero if

$$A_c < \overline{L}^{-\alpha}\left(\Omega^{\sigma-1} - \left(\frac{\kappa A_r}{\tau}\right)^{\sigma-1}\right)^{\frac{1}{\sigma-1}}$$

which is satisfied under the initial assumption given that it is the initial assumption.

(ii) **Case #2:** $\left(\Omega^{\sigma-1} - \left(\frac{\kappa A_r}{\tau}\right)^{\sigma-1}\right)^{\frac{1}{\sigma-1}}$ is the upper bound and evaluating the limit of $L_c(r,c)$ as w_c approaches $\left(\Omega^{\sigma-1} - \left(\frac{\kappa A_r}{\tau}\right)^{\sigma-1}\right)^{\frac{1}{\sigma-1}}$ reveals:

$$\lim_{w_c \to \xi} L_c(r,c) = A_c^{-\frac{1}{\alpha}}\left(\Omega^{\sigma-1} - \left(\frac{\kappa A_r}{\tau}\right)^{\sigma-1}\right)^{\frac{1}{\alpha(\sigma-1})} - \Omega^{\frac{\delta}{1-\delta}}\left(\frac{H_c}{1-\delta}\right)\left(\frac{B_c}{\overline{V}}\right)^{\frac{1}{1-\delta}}$$

where $\xi = \left(\Omega^{\sigma-1} - \left(\frac{\kappa A_r}{\tau}\right)^{\sigma-1}\right)^{\frac{1}{\sigma-1}}$. This limit is greater than zero if

$$A_c < \overline{L}^{-\alpha}\left(\Omega^{\sigma-1} - \left(\frac{\kappa A_r}{\tau}\right)^{\sigma-1}\right)^{\frac{1}{\sigma-1}}$$

which contradicts the initial assumption.

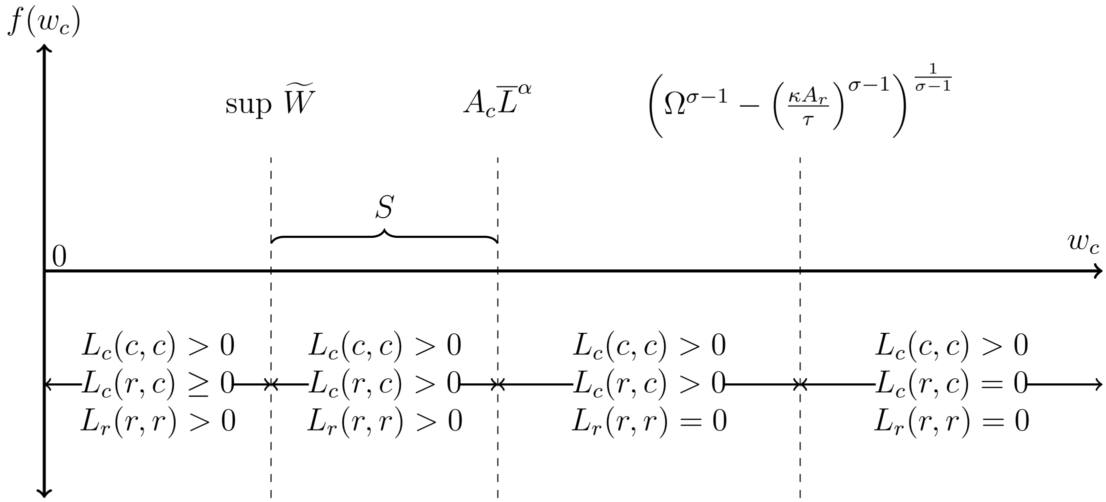
Given that case #2 results in a contradiction, only under the assumption in case #1 that

$$A_c < \overline{L}^{-\alpha}\left(\Omega^{\sigma-1} - \left(\frac{\kappa A_r}{\tau}\right)^{\sigma-1}\right)^{\frac{1}{\sigma-1}}$$

ensures satisfaction of Bolzano's intermediate value theorem. The above is condition (2.9) in Lemma 2. Bolzano's intermediate value theorem then implies the existence of a set $\widetilde{W}$ comprised of urban wages $\widetilde{w_c} \in (0, A_c\overline{L}^{\alpha})$ at which $L_c(r,c)(\widetilde{w_c}) = 0$. Denoting the supremum of this set $\sup \widetilde{W}$ and noting that $L_c(r,c) > 0$ for any $w_c > \sup \widetilde{W}$, define a set $S = (\sup \widetilde{W}, A_c\overline{L}^{\alpha})$. It follows for any $w_c \in S$, the values of $L_c(c,c)$, $L_c(r,c)$, and $L_r(r,r)$ are all strictly positive in equilibrium, implying all worker types exist in equilibrium. ■

In Figure A.2, I sketch the location of set S in $(w_c, f(w_c))$-space. Important thresholds in the proof on the w_c-axis are demarcated by dashed lines. These thresholds delineate zones along the w_c-axis, within which result in different sizes for the various worker types, which I record below the horizontal axis in each zone. The set S, laying between $\sup \widetilde{W}$ and $A_c\overline{L}^{\alpha}$ exclusive, is the only zone for which all worker types are strictly positive.

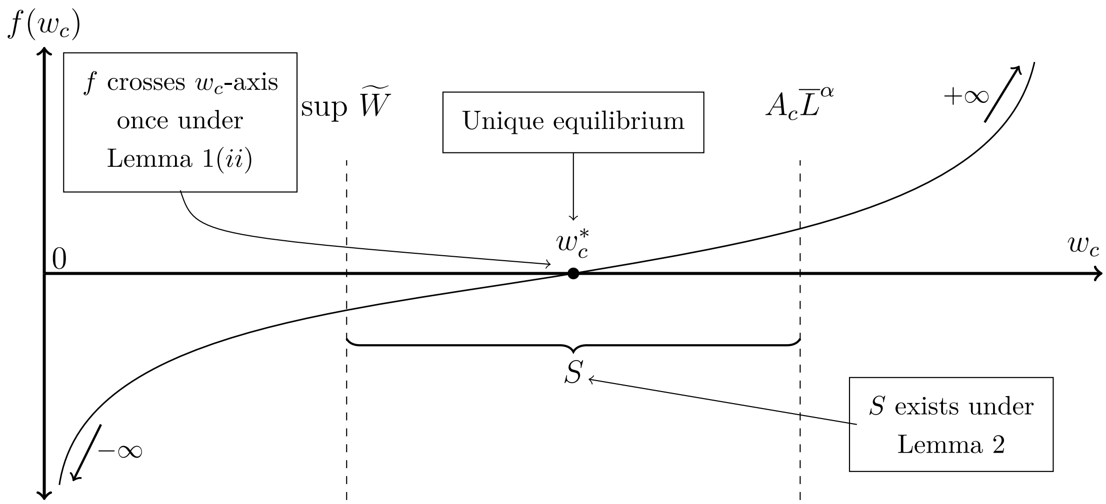
Figure A.2: Existence of the Set S under Lemma 2



Notes: This figure identifies boundaries and the set of interest, S, that arise under the conditions set forth in Lemma 2. The w_c-axis represents the strictly positive wages $w_c \in \mathbb{R}_{++}$ earned by workers in the city, while the y-axis represents the values f can take on when evaluated at w_c, $f(w_c) \in \mathbb{R}$. The important thresholds along the w_c-axis are labelled with dashed lines, which induce the formation of four "zones" along the horizontal axis. The existence of the various populations are specified below the horizontal axis. The only zone in which all worker types exist is the set S.

A.5 Sketch of Proposition 1

Figure A.3: Existence and Uniqueness of an Equilibrium under Proposition 1



Notes: This figure illustrates what a unique equilibrium might look like under Proposition 1, which assumes Lemmas 1(ii) and 2 and that the unique solution to 2.6 guaranteed to exist under 1(ii) is an element of the set S from Lemma 2. Define the function f as the LHS of Equation 2.6 subtracted from the RHS of 2.6, where a solution to 2.6 is defined as a horizontal axis crossing point of f. The (horizontal) w_c-axis represents the strictly positive wages $w_c \in \mathbb{R}_{++}$ earned by workers in the city, while the (vertical) $f((w_c)$-axis represents the values f can take on when evaluated at w_c, $f(w_c) \in \mathbb{R}$. The arrows in the bottom left and top right indicate the behaviour of f as w_c approaches zero from the right and positive infinity, respectively.

A.6 Hat Algebra about the Unique Equilibrium

A.6.1 Log-linearisation

Taking the natural log of both sides of equation (2.6) setting $\sigma = \frac{1}{1-\delta}$ yields:

$$\ln\left(A_r\overline{L} + \left(\frac{w_c}{A_c}\right)^{\frac{1}{\alpha}}(w_c - A_r)\right) = \tag{A.35}$$

$$\ln\left(\frac{\delta}{1-\delta}\right) - \left(\frac{1}{1-\delta}\right)\ln(\overline{V})$$

$$+\ln\left(B_c^{\frac{1}{1-\delta}}H_c\left(w_c^{\frac{1}{1-\delta}} + \left(\frac{\kappa A_r}{\tau}\right)^{\frac{1}{1-\delta}}\right) + \frac{B_r^{\frac{1}{1-\delta}}H_r}{\kappa^{\frac{1}{1-\delta}}}\left(\left(\frac{w_c}{\tau}\right)^{\frac{1}{1-\delta}} + (\kappa A_r)^{\frac{1}{1-\delta}}\right)\right)$$

Using the system of equations (A.6) through (A.19) that summarise this model, certain components of equation A.35 can be reexpressed as follows:

$$A_r\overline{L} + \left(\frac{w_c}{A_c}\right)^{\frac{1}{\alpha}}(w_c - A_r) = Y_r + Y_c \tag{A.36}$$

$$B_c^{\frac{1}{1-\delta}} H_c\Big(w_c^{\frac{1}{1-\delta}} + \left(\frac{\kappa A_r}{\tau}\right)^{\frac{1}{1-\delta}}\Big) + \frac{B_r^{\frac{1}{1-\delta}} H_r}{\kappa^{\frac{1}{1-\delta}}}\Big(\left(\frac{w_c}{\tau}\right)^{\frac{1}{1-\delta}} + (\kappa A_r)^{\frac{1}{1-\delta}}\Big) \tag{A.37}$$
$$= \left(\frac{1-\delta}{\delta}\right)\overline{V}^{\frac{1}{1-\delta}}\Big(X_c + X_r\Big)$$

Additionally, applying similar algebraic transformations in the derivation of equation (A.28), equilibrium total demand for the urban and rural good can be expressed:

$$X_c = \left(\frac{\delta}{1-\delta}\right)\left(\frac{w_c}{\overline{V}}\right)^{\frac{1}{1-\delta}}\left(B_c^{\frac{1}{1-\delta}} H_c + \frac{B_r^{\frac{1}{1-\delta}} H_r}{(\kappa\tau)^{\frac{1}{1-\delta}}}\right) \tag{A.38}$$

$$X_r = \left(\frac{\delta}{1-\delta}\right)\left(\frac{w_c}{\overline{V}}\right)^{\frac{1}{1-\delta}}\left(\frac{B_c^{\frac{1}{1-\delta}} H_r \kappa^{\frac{1}{1-\delta}}}{\tau^{\frac{1}{1-\delta}}} + B_r^{\frac{1}{1-\delta}} H_r\right) \tag{A.39}$$

Finally, via equations (A.25) and (A.27) respectively, the products $B_c^{\frac{1}{1-\delta}} H_c$ and $B_r^{\frac{1}{1-\delta}} H_r$ can be written in equilibrium as:

$$B_c^{\frac{1}{1-\delta}} H_c = \frac{(1-\delta) P_c^{\frac{\delta}{1-\delta}} \overline{V}^{\frac{1}{1-\delta}}}{w_c^{\frac{\delta}{1-\delta}}} L_c(c,c) \tag{A.40}$$

$$B_r^{\frac{1}{1-\delta}} H_r = \frac{(1-\delta)\kappa^{\frac{\delta}{1-\delta}} P_r^{\frac{\delta}{1-\delta}} \overline{V}^{\frac{1}{1-\delta}}}{w_c^{\frac{\delta}{1-\delta}}}\Big(L_r(r,r) + L_c(r,c)\Big) \tag{A.41}$$

Let $\overline{L}_{A_c} = \partial\overline{L}/\partial A_c$. Totally differentiating equation (A.35) and substituting equations (A.36) through (A.41) where applicable reveals:

$$\left(\frac{(\frac{1+\alpha}{\alpha})Y_c - A_r L_c}{Y_c + Y_r}\right)\widehat{w_c} - \left(\frac{(\frac{1}{\alpha})\Big(Y_c - A_r L_c\Big) - A_c A_r \overline{L}_{A_c}}{Y_c + Y_r}\right)\widehat{A_c} + \left(\frac{A_r L_c + A_r^2 \overline{L}_{A_r}}{Y_c + Y_r}\right)\widehat{A_r}$$
$$+ \left(\frac{B_c A_r \overline{L}_{B_c}}{Y_c + Y_r}\right)\widehat{B_c} + \left(\frac{B_r A_r \overline{L}_{B_r}}{Y_c + Y_r}\right)\widehat{B_r} = \left(\frac{(\frac{1}{1-\delta})X_c}{X_c + X_r}\right)\widehat{w_c} + \left(\frac{(\frac{1}{1-\delta})X_r}{X_c + X_r}\right)\widehat{A_r}$$
$$- \left(\frac{1}{1-\delta}\right)\widehat{\overline{V}} + \left(\frac{\delta\Big(\Big(1 + (\tau p_r)^{-\frac{1}{1-\delta}}\Big)/P_c^{-\frac{\delta}{1-\delta}}\Big)w_c L_c(c,c)}{X_c + X_r}\right)\left(\left(\frac{1}{1-\delta}\right)\widehat{B_c} + \widehat{H_c}\right)$$
$$+ \left(\frac{\delta\Big(\Big(\tau^{-\frac{1}{1-\delta}} + p_r^{-\frac{1}{1-\delta}}\Big)/P_r^{-\frac{\delta}{1-\delta}}\Big)w_r\Big(L_r(r,r) + L_c(r,c)\Big)}{X_c + X_r}\right)\left(\left(\frac{1}{1-\delta}\right)\widehat{B_r} + \widehat{H_r}\right)$$
$$+ \left(\frac{\left(\frac{\delta}{1-\delta}\right)\left(\Big((\tau p_r)^{-\frac{1}{1-\delta}}/P_c^{-\frac{\delta}{1-\delta}}\Big)w_c L_c(c,c) - \Big(\tau^{-\frac{1}{1-\delta}}/P_r^{-\frac{\delta}{1-\delta}}\Big)w_r\Big(L_r(r,r) + L_c(r,c)\Big)\right)}{X_c + X_r}\right)\widehat{\kappa}$$
$$- \left(\frac{\left(\frac{\delta}{1-\delta}\right)\left(\Big((\tau p_r)^{-\frac{1}{1-\delta}}/P_c^{-\frac{\delta}{1-\delta}}\Big)w_c L_c(c,c) + \Big(\tau^{-\frac{1}{1-\delta}}/P_r^{-\frac{\delta}{1-\delta}}\Big)w_r\Big(L_r(r,r) + L_c(r,c)\Big)\right)}{X_c + X_r}\right)\widehat{\tau} \tag{A.42}$$

where $\widehat{x} = \frac{dx}{x}$ for $x = \{w_c, A_c, A_r, B_c, B_r, H_c, H_r, \kappa, \tau, \overline{V}\}$, with $\widehat{x}$ denoting a (small) proportional change in x á la the familiar Jones (1965) "hat algebra." By equation (2.5), in equilibrium total demand for the good produced in i but consumed in $i' \neq i$, denoted $X_{ii'}$, can be written as:

$$X_{rc} = \delta\Big((\tau p_r)^{-\frac{1}{1-\delta}}/P_c^{-\frac{\delta}{1-\delta}}\Big)w_c L_c(c,c) \tag{A.43}$$

$$X_{cr} = \delta\Big(\tau^{-\frac{1}{1-\delta}}/P_r^{-\frac{\delta}{1-\delta}}\Big)w_r\Big(L_r(r,r) + L_c(r,c)\Big) \tag{A.44}$$

Similarly, equilibrium total demand for the good produced and consumed in i, denoted X_{ii}, can be expressed:

$$X_{cc} = \delta\Big(1/P_c^{-\frac{\delta}{1-\delta}}\Big)w_c L_c(c,c) \tag{A.45}$$

$$X_{rr} = \delta\Big(p_r^{-\frac{1}{1-\delta}}/P_r^{-\frac{\delta}{1-\delta}}\Big)w_r\Big(L_r(r,r) + L_c(r,c)\Big) \tag{A.46}$$

Since in equilibrium $Y_c + Y_r = X_c + X_r$, multiplying both sides of equation (A.42) $(Y_c + Y_r)$, substituting equations (A.43) through (A.46) where applicable, substituting Y_c for X_c in the expression multiplying $\widehat{w_c}$ on the right hand side given $X_c = Y_c$ in equilibrium, and solving for $\widehat{w_c}$ yields:

$$
\begin{aligned}
\widehat{w_c} = & \left[\frac{(\frac{1}{\alpha})(A_c L_c^{\alpha} - A_r)L_c - \beta_{\overline{L}A_c} A_r \overline{L}}{\left(\left(\frac{1-\delta(1+\alpha)}{\alpha(1-\delta)}\right) A_c L_c^{\alpha} - A_r\right) L_c}\right] \widehat{A_c} + \left[\frac{\left((\frac{1}{1-\delta})L_r - L_c - \beta_{\overline{L}A_r} A_r \overline{L}\right) A_r}{\left(\left(\frac{1-\delta(1+\alpha)}{\alpha(1-\delta)}\right) A_c L_c^{\alpha} - A_r\right) L_c}\right] \widehat{A_r} \\
& + \left[\frac{(\frac{1}{1-\delta})(X_{cc} + X_{rc}) - \beta_{\overline{L}B_c} A_r \overline{L}}{\left(\left(\frac{1-\delta(1+\alpha)}{\alpha(1-\delta)}\right) A_c L_c^{\alpha} - A_r\right) L_c}\right] \widehat{B_c} + \left[\frac{(\frac{1}{1-\delta})(X_{cr} + X_{rr}) - \beta_{\overline{L}B_r} A_r \overline{L}}{\left(\left(\frac{1-\delta(1+\alpha)}{\alpha(1-\delta)}\right) A_c L_c^{\alpha} - A_r\right) L_c}\right] \widehat{B_r} \\
& + \left[\frac{X_{cc} + X_{rc}}{\left(\left(\frac{1-\delta(1+\alpha)}{\alpha(1-\delta)}\right) A_c L_c^{\alpha} - A_r\right) L_c}\right] \widehat{H_c} + \left[\frac{X_{cr} + X_{rr}}{\left(\left(\frac{1-\delta(1+\alpha)}{\alpha(1-\delta)}\right) A_c L_c^{\alpha} - A_r\right) L_c}\right] \widehat{H_r} \\
& + \left[\frac{(\frac{1}{1-\delta})(X_{rc} - X_{cr})}{\left(\left(\frac{1-\delta(1+\alpha)}{\alpha(1-\delta)}\right) A_c L_c^{\alpha} - A_r\right) L_c}\right] \widehat{\kappa} + \left[\frac{(\frac{1}{1-\delta})(X_{rc} + X_{cr})}{\left(A_r - \left(\frac{1-\delta(1+\alpha)}{\alpha(1-\delta)}\right) A_c L_c^{\alpha}\right) L_c}\right] \widehat{\tau} \\
& + \left[\frac{(\frac{1}{1-\delta})(Y_c + Y_r)}{\left(A_r - \left(\frac{1-\delta(1+\alpha)}{\alpha(1-\delta)}\right) A_c L_c^{\alpha}\right) L_c}\right] \widehat{\overline{V}} \\
= & \beta_{w_c A_c} \widehat{A_c} + \beta_{w_c A_r} \widehat{A_r} + \beta_{w_c B_c} \widehat{B_c} + \beta_{w_c B_r} \widehat{B_r} + \beta_{w_c H_c} \widehat{H_c} + \beta_{w_c H_r} \widehat{H_r} \\
& + \beta_{w_c \kappa} \widehat{\kappa} + \beta_{w_c \tau} \widehat{\tau} + \beta_{w_c \overline{V}} \widehat{\overline{V}}
\end{aligned}
\tag{A.47}
$$

where $\beta_{w_c x}$ and $\beta_{\overline{L}x}$ are the elasticities of the urban wage and region's population with respect to exogenous parameters $x = \{\tau, \kappa, \overline{V}, B_i, A_i, H_i\}$ for $i \in \{c, r\}$.

Applying logarithmic transformations to equations (A.25), (A.26), (A.29), and (A.30), totally differentiating, and substituting equation (A.47) for $\widehat{w_c}$ reveals:

$$
\begin{aligned}
\widehat{w_r} = & \beta_{w_c A_c} \widehat{A_c} + \beta_{w_c A_r} \widehat{A_r} + \beta_{w_c B_c} \widehat{B_c} + \beta_{w_c B_r} \widehat{B_r} + \beta_{w_c H_c} \widehat{H_c} + \beta_{w_c H_r} \widehat{H_r} \\
& + (\beta_{w_c \kappa} - 1) \widehat{\kappa} + \beta_{w_c \tau} \widehat{\tau} + \beta_{w_c \overline{V}} \widehat{\overline{V}}
\end{aligned}
\tag{A.48}
$$

$$
\begin{aligned}
\widehat{L_c} = & \tfrac{1}{\alpha}(\beta_{w_c A_c} - 1) \widehat{A_c} + \tfrac{1}{\alpha} \beta_{w_c A_r} \widehat{A_r} + \tfrac{1}{\alpha} \beta_{w_c B_c} \widehat{B_c} + \tfrac{1}{\alpha} \beta_{w_c B_r} \widehat{B_r} + \tfrac{1}{\alpha} \beta_{w_c H_c} \widehat{H_c} \\
& + \tfrac{1}{\alpha} \beta_{w_c H_r} \widehat{H_r} + \tfrac{1}{\alpha} \beta_{w_c \kappa} \widehat{\kappa} + \tfrac{1}{\alpha} \beta_{w_c \tau} \widehat{\tau} + \tfrac{1}{\alpha} \beta_{w_c \overline{V}} \widehat{\overline{V}}
\end{aligned}
\tag{A.49}
$$

$$\widehat{L_c(c,c)} = \left[\left(\frac{\delta}{1-\delta}\right)\frac{\beta_{w_c A_c}}{P_c^{-\frac{\delta}{1-\delta}}}\right]\widehat{A_c} + \left[\left(\frac{\delta}{1-\delta}\right)\left(\frac{(\tau p_r)^{-\frac{\delta}{1-\delta}} + \beta_{w_c A_r}}{P_c^{-\frac{\delta}{1-\delta}}}\right)\right]\widehat{A_r}$$
$$+ \left[\left(\frac{1}{1-\delta}\right)\left(1 + \frac{\delta\beta_{w_c B_c}}{P_c^{-\frac{\delta}{1-\delta}}}\right)\right]\widehat{B_c} + \left[\left(\frac{\delta}{1-\delta}\right)\frac{\beta_{w_c B_r}}{P_c^{-\frac{\delta}{1-\delta}}}\right]\widehat{B_r}$$
$$+ \left[1 + \left(\frac{\delta}{1-\delta}\right)\frac{\beta_{w_c H_c}}{P_c^{-\frac{\delta}{1-\delta}}}\right]\widehat{H_c} + \left[\left(\frac{\delta}{1-\delta}\right)\frac{\beta_{w_c H_r}}{P_c^{-\frac{\delta}{1-\delta}}}\right]\widehat{H_r} \tag{A.50}$$
$$+ \left[\left(\frac{\delta}{1-\delta}\right)\left(\frac{(\tau p_r)^{-\frac{\delta}{1-\delta}} + \beta_{w_c \kappa}}{P_c^{-\frac{\delta}{1-\delta}}}\right)\right]\widehat{\kappa} + \left[\left(\frac{\delta}{1-\delta}\right)\left(\frac{\beta_{w_c \tau} - (\tau p_r)^{-\frac{\delta}{1-\delta}}}{P_c^{-\frac{\delta}{1-\delta}}}\right)\right]\widehat{\tau}$$
$$+ \left[\left(\frac{1}{1-\delta}\right)\left(\frac{\delta\beta_{w_c \overline{V}}}{P_c^{-\frac{\delta}{1-\delta}}} - 1\right)\right]\widehat{\overline{V}}$$

$$\widehat{L_c(r,c)} = \left[\frac{\left(1 - \mu_{cc}^{L_c}(\frac{\delta}{1-\delta})(\frac{1}{P_c})^{-\frac{\delta}{1-\delta}}\right)\beta_{w_c A_c} - 1}{\alpha\mu_{rc}^{L_c}}\right]\widehat{A_c}$$
$$+ \left[\frac{\left(1 - \mu_{cc}^{L_c}(\frac{\delta}{1-\delta})(\frac{1}{P_c})^{-\frac{\delta}{1-\delta}}\right)\beta_{w_c A_r} - \mu_{cc}^{L_c}(\frac{\delta}{1-\delta})(\frac{\tau p_r}{P_c})^{-\frac{\delta}{1-\delta}}}{\alpha\mu_{rc}^{L_c}}\right]\widehat{A_r}$$
$$+ \left[\frac{\left(1 - \mu_{cc}^{L_c}(\frac{\delta}{1-\delta})(\frac{1}{P_c})^{-\frac{\delta}{1-\delta}}\right)\beta_{w_c B_c} - \mu_{cc}^{L_c}(\frac{1}{1-\delta})}{\alpha\mu_{rc}^{L_c}}\right]\widehat{B_c}$$
$$+ \left[\frac{\left(1 - \mu_{cc}^{L_c}(\frac{\delta}{1-\delta})(\frac{1}{P_c})^{-\frac{\delta}{1-\delta}}\right)\beta_{w_c B_r}}{\alpha\mu_{rc}^{L_c}}\right]\widehat{B_r}$$
$$+ \left[\frac{\left(1 - \mu_{cc}^{L_c}(\frac{\delta}{1-\delta})(\frac{1}{P_c})^{-\frac{\delta}{1-\delta}}\right)\beta_{w_c H_c} - \mu_{cc}^{L_c}}{\alpha\mu_{rc}^{L_c}}\right]\widehat{H_c} \tag{A.51}$$
$$+ \left[\frac{\left(1 - \mu_{cc}^{L_c}(\frac{\delta}{1-\delta})(\frac{1}{P_c})^{-\frac{\delta}{1-\delta}}\right)\beta_{w_c H_r}}{\alpha\mu_{rc}^{L_c}}\right]\widehat{H_r}$$
$$+ \left[\frac{\left(1 - \mu_{cc}^{L_c}(\frac{\delta}{1-\delta})(\frac{1}{P_c})^{-\frac{\delta}{1-\delta}}\right)\beta_{w_c \kappa} - \mu_{cc}^{L_c}(\frac{\delta}{1-\delta})(\frac{\tau p_r}{P_c})^{-\frac{\delta}{1-\delta}}}{\alpha\mu_{rc}^{L_c}}\right]\widehat{\kappa}$$
$$+ \left[\frac{\left(1 - \mu_{cc}^{L_c}(\frac{\delta}{1-\delta})(\frac{1}{P_c})^{-\frac{\delta}{1-\delta}}\right)\beta_{w_c \tau} + \mu_{cc}^{L_c}(\frac{\delta}{1-\delta})(\frac{\tau p_r}{P_c})^{-\frac{\delta}{1-\delta}}}{\alpha\mu_{rc}^{L_c}}\right]\widehat{\tau}$$
$$+ \left[\frac{\left(1 - \mu_{cc}^{L_c}(\frac{\delta}{1-\delta})(\frac{1}{P_c})^{-\frac{\delta}{1-\delta}}\right)\beta_{w_c \overline{V}} - \mu_{cc}^{L_c}(\frac{1}{1-\delta})}{\alpha\mu_{rc}^{L_c}}\right]\widehat{\overline{V}}$$

$$\widehat{L_r} = \widehat{L_r(r,r)} = \underbrace{\left[\frac{\beta_{\overline{L}A_c} + \mu_c(1 - \frac{1}{\alpha}\beta_{w_cA_c})}{\mu_r}\right]}_{\beta_{L_rA_c}} \widehat{A_c} + \left[\frac{\alpha\beta_{\overline{L}A_r} - \mu_c\beta_{w_cA_r}}{\alpha\mu_r}\right] \widehat{A_r}$$
$$+ \left[\frac{\alpha\beta_{\overline{L}B_c} - \mu_c\beta_{w_cB_c}}{\alpha\mu_r}\right] \widehat{B_c} + \left[\frac{\alpha\beta_{\overline{L}B_r} - \mu_c\beta_{w_cB_r}}{\alpha\mu_r}\right] \widehat{B_r} + \left[\frac{-\mu_c\beta_{w_cH_c}}{\alpha\mu_r}\right] \widehat{H_c}$$
$$+ \left[\frac{-\mu_c\beta_{w_cH_r}}{\alpha\mu_r}\right] \widehat{H_r} + \left[\frac{-\mu_c\beta_{w_c\kappa}}{\alpha\mu_r}\right] \widehat{\kappa} + \left[\frac{-\mu_c\beta_{w_c\tau}}{\alpha\mu_r}\right] \widehat{\tau} + \left[\frac{-\mu_c\beta_{w_c\overline{V}}}{\alpha\mu_r}\right] \widehat{\overline{V}} \tag{A.52}$$

where the various μ represent labour shares, specifically:

$$\mu_{cc}^{L_c} = \frac{L_c(c,c)}{L_c}$$
$$\mu_{rc}^{L_c} = \frac{L_c(r,c)}{L_c}$$
$$\mu_c = \frac{L_c}{\overline{L}}$$
$$\mu_r = \frac{L_r}{\overline{L}}$$

A.6.2 Deriving the Reduced-Form Relationship of Interest (Equation 2.8)

Isolating the parameter $\beta_{w_cA_c}$ in equation (A.47) and substituting $\eta = (1+\alpha)/\alpha - 1/(1-\delta)$, we can rewrite $\beta_{w_cA_c}$ as:

$$\beta_{w_cA_c} = \frac{(\frac{\mu_c}{\alpha})A_cL_c^\alpha - \left(\frac{\mu_c + \alpha\beta_{\overline{L}A_c}}{\mu_c}\right)A_r}{\mu_c(\eta A_cL_c^\alpha - A_r)}$$

Substituting the above into $\beta_{L_rA_c}$ from equation (A.52) and performing algebraic manipulation yields:

$$\begin{aligned}
\beta_{L_rA_c} =& \frac{\beta_{\overline{L}A_c}}{\mu_r} + \frac{\mu_c}{\mu_r}\left(1 - \frac{\beta_{w_cA_c}}{\alpha}\right) \\
=& \frac{\beta_{\overline{L}A_c}}{\mu_r} + \frac{\mu_c}{\mu_r}\left(1 - \frac{1}{\alpha}\left(\frac{(\frac{\mu_c}{\alpha})A_cL_c^\alpha - \left(\frac{\mu_c + \alpha\beta_{\overline{L}A_c}}{\mu_c}\right)A_r}{\mu_c(\eta A_cL_c^\alpha - A_r)}\right)\right) \\
=& \frac{(1/(\alpha^2\eta) - 1)\mu_c - \beta_{\overline{L}A_c}}{\mu_r}\left[\frac{\left(\frac{1-\alpha}{\alpha^2}\right)\left(\frac{(1+\alpha)\mu_c + \alpha\beta_{\overline{L}A_c}}{(1/(\alpha^2\eta)-1)\mu_c - \beta_{\overline{L}A_c}}\right) - (A_cL_c^\alpha/A_r)}{(A_cL_c^\alpha/A_r) - \eta^{-1}}\right] \\
=& \Theta_1\left[\frac{\Theta_2 - (A_cL_c^\alpha/A_r)}{(A_cL_c^\alpha/A_r) - \Theta_3}\right]
\end{aligned}$$

where again

$$\Theta_1 = \frac{(1/(\alpha^2\eta) - 1)\mu_c - \beta_{\overline{L}A_c}}{\mu_r}$$
$$\Theta_2 = \left(\frac{1-\alpha}{\alpha^2}\right)\left(\frac{(1+\alpha)\mu_c + \alpha\beta_{\overline{L}A_c}}{(1/(\alpha^2\eta) - 1)\mu_c - \beta_{\overline{L}A_c}}\right)$$
$$\Theta_3 = \frac{1}{\eta}$$

A.7 Proof of Proposition 2

Proof. The assumptions under Proposition 1 and the assumption on the size of $\beta_{\overline{L}A_c}$ ensure Θ_1, Θ_2, and Θ_3 are strictly positive in equilibrium.

Conditions for $\Theta_1 > 0$. For Θ_1 to be strictly positive, we require 1) that $(1/(\alpha^2\eta) - 1) > 0$ and 2) that $\beta_{\overline{L}A_c} < (1/(\alpha^2\eta) - 1)\mu_c$. The former is ensured by the assumption that $\sigma > 1$. The product $\alpha^2\eta$ may be expressed as:

$$\alpha^2\eta = \alpha^2\left(\frac{1+\alpha}{\alpha} - \frac{1}{1-\delta}\right) = \alpha^2\left(\frac{1+\alpha}{\alpha} - \sigma\right) = \alpha\left(1 + \alpha - \alpha\sigma\right)$$

Thus, it follows that:

$$\begin{aligned}(1/(\alpha^2\eta) - 1) > 0 &\iff 1/(\alpha^2\eta) > 1\\ &\iff \frac{1}{\alpha} > 1 + \alpha - \alpha\sigma\\ &\iff \sigma > 1\end{aligned}$$

which holds by my initial assumption on σ. The latter is guaranteed by $\beta_{\overline{L}A_c} < (1/(\alpha^2\eta)-1)\mu_c$. Therefore, $\Theta_1 > 0$.

Conditions for $\Theta_2 > 0$. For Θ_2, we require that 1) $\alpha < 1$, 2) $(1/(\alpha^2\eta) - 1) > 0$, and 3) $\beta_{\overline{L}A_c} < (1/(\alpha^2\eta) - 1)\mu_c$. The first requirement is satisfied by my initial assumption on α and the other two requirements are satisfied by proof of $\Theta > 0$.

Conditions for $\Theta_3 > 0$. For Θ_3 to be strictly positive, it must be that $\eta > 0$:

$$\eta = \left(\frac{1+\alpha}{\alpha} - \frac{1}{1-\delta}\right) > 0 \iff \frac{1+\alpha}{\alpha} > \frac{1}{1-\delta}$$

which is true by assumption in Proposition 1. Assuming that $A_c L_c^\alpha / A_r > \max\{\Theta_2, \Theta_3\}$, it follows that:

$$\beta_{L_r A_c} = \underbrace{\Theta_1}_{>0} \left[\underbrace{\left(\Theta_2 - \frac{A_c L_c^\alpha}{A_r}\right)}_{<0} \Big/ \underbrace{\left(\frac{A_c L_c^\alpha}{A_r} - \Theta_3\right)}_{>0} \right] < 0$$

■

A.8 Back of the Envelope Proposition 2 Calibration

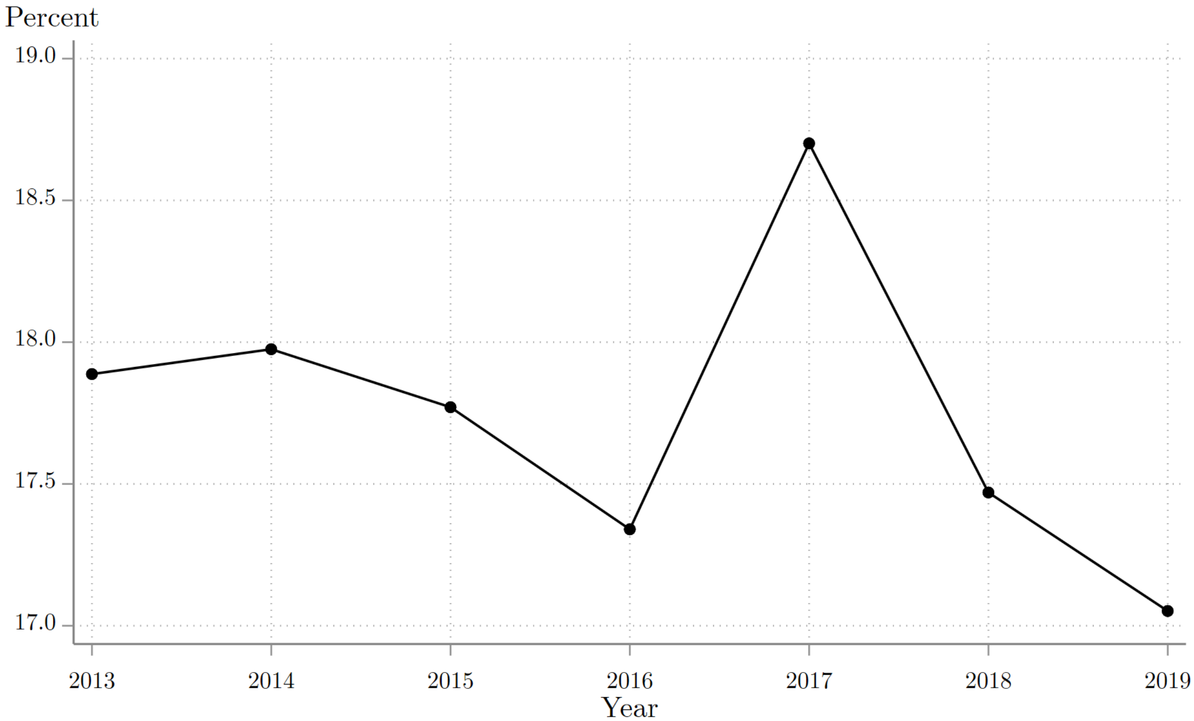
To get a rough feel for the sizes of the bounds in Proposition 2, I calibrate exogenous parameters in the model using external data, values in the literature, and the structural requirements set forth in Proposition 1.

Preferences and Agglomeration (δ, σ, and α). The Bureau of Labor Statistics (BLS) publish annual expenditure breakdowns of the average American household in the Consumption Expenditure (CE) Survey (BLS, 2021). Dividing the average annual expenditure on shelter (i.e., mortgage payments and rents in the CE) by average annual after-tax income, I plot the percent of household income going towards housing from 2013 to 2019 in Figure A.4. American households spend approximately 20% of their after-tax income on housing each year. Taking this fact to this model, I calibrate $1 - \delta = 0.2$. By Lemma 1(ii), $1/(1-\delta) = \sigma$, implying if $1 - \delta = 0.2$, $\sigma = 5$. An elasticity of substitution equal to 5 lands very close to calibrations common in the spatial literature. For instance Allen and Arkolakis (2014) and Redding (2016) calibrate $\sigma = 4$ in their quantitative spatial models while most numerical core-periphery analyses in Fujita, Krugman, and Venables (1999) set $\sigma = 5$. Lemma 1(ii) also requires that $\sigma \in \left(\frac{1}{\alpha}, \frac{1+\alpha}{\alpha}\right)$. Thus, if $\sigma = 5$, $\alpha \in (0.2, 0.25)$. Most of the literature on agglomeration spillovers, such as Rosenthal and Strange (2004), implies α is small, so I choose a value of α close to the lower bound, say $\alpha = 0.21$.

Urban Population Share (μ_c). The Bureau of Economic Analysis (BEA) publish annual county population estimates (BEA, 2021). By first grouping counties into "rural" and "urban" classifications based upon their Rural-Urban Continuum Code (RUCC) designated by the United States Department of Agriculture (USDA) Economic Research Service (ERS) (ERS, 2020), I aggregate the population of US residents living in urban counties and dividing that value by the total US population, which yields the US urban population share. I plot the evolution of this share, as well as the share of US residents living in rural counties, from 2013 to 2019 in Figure A.5. Over this period of observation, approximately 85% of the US population resides in a county designated as "urban." As such, I set $\mu_c = 0.85$.

Urban TFP Extra-Regional In-Migration Elasticity ($\beta_{\overline{L}A_c}$). Hornbeck and Moretti (2021) present a method for estimating the number of workers moving to a city in response to local manufacturing TFP growth. Using data on TFP growth in cities around the U.S. from 1980 to 1990, they show how their estimation method can estimate the number of workers

Figure A.4: Average Household Housing Expenditure Share, 2013-2019



Source: Bureau of Labor Statistics

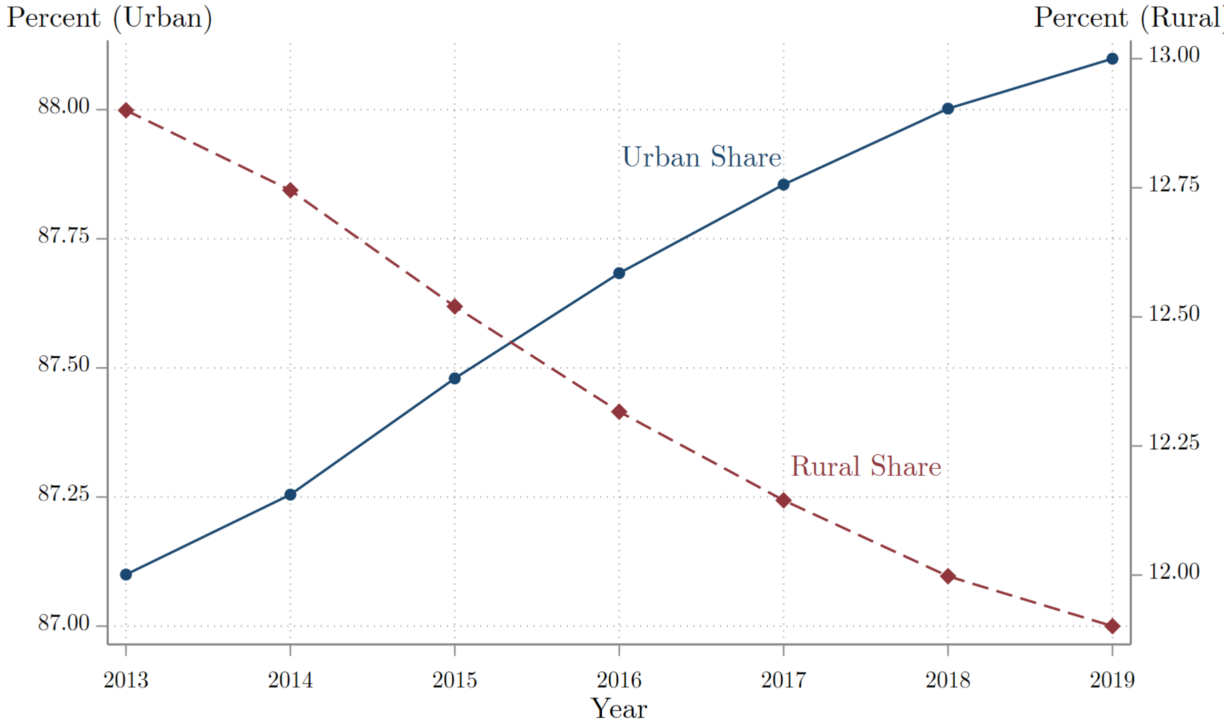
Notes: This figure plots the annual mean share of household after-tax income spent on housing (mortgages or rent) in the U.S. from 2013 to 2019. The data are sourced from the BLS Consumer Expenditure Survey (BLS, 2021).

moving from other cities in the U.S. from 1980 to 2000 directly in response to this TFP growth. They use three cities as examples: Houston, San Jose, and Cincinnati. I use the results of these cities in combination with initial employment data from the 1980 Quarterly Census of Employment and Wages (QCEW) publicly available through the BLS to derive a ballpark calibration for $\beta_{\overline{L}A_c}$.

Houston. In response to its 2.4% TFP growth, Hornbeck and Moretti (2021) estimate that, on average, 291 workers moved from another city in the U.S. to Houston. Given there are 193 Metropolitan Statistical Areas (MSAs) in their sample, multiplying this amount by the city average, the total in-migration to Houston was roughly 56,163 workers from 1980 to 2000. According to the QCEW, in January of 1980, 1,172,259 people were employed in Houston, implying local employment grew 4.7%. It follows that:

$$\beta_{\overline{L}A_c}^{\text{HOU}} = \frac{\partial \overline{L}}{\partial A_c}\frac{A_c}{\overline{L}} = \frac{4.7\%}{2.4\%} = 2.0$$

Figure A.5: Urban and Rural U.S. Employment Shares, 2013-2019



Source: Bureau of Economic Analysis and U.S. Department of Agriculture Economic Research Service

Notes: This figure plots the annual urban and rural employment shares in the U.S. from 2013 to 2019. The "urban" employment total is the number of employees reported by BEA (2021) as working in a county designated as urban by ERS (2020), while the rural employment total is the number of employees working in a rural (nonurban) county. Dividing these totals by the total U.S. workforce count for each year yields the series plotted above.

San Jose. On account of 16.4% manufacturing TFP growth, San Jose saw 272,709 new workers move to the city (= 1,413 new workers from other cities on average * 193 MSAs). The initial 1980 QCEW employment level was 579,752, implying the San Jose employment grew 47% from 1980 to 2000 on account of the TFP shock. Thus:

$$\beta_{\overline{L}A_c}^{\text{SJ}} = \frac{\partial \overline{L}}{\partial A_c}\frac{A_c}{\overline{L}} = \frac{47.0\%}{16.4\%} = 2.9$$

Cincinnati. Cincinnati's 2.0% manufacturing TFP growth stimulated 16,212 workers to move from elsewhere (= average of 84 workers coming from other cities * 193 MSAs). Since the initial 1980 QCEW employment count was 501,985, TFP growth resulted in 3.2% employment growth via in-migration. Therefore:

$$\beta_{\overline{L}A_c}^{\text{CIN}} = \frac{\partial \overline{L}}{\partial A_c}\frac{A_c}{\overline{L}} = \frac{3.2\%}{2.0\%} = 1.6$$

Table A.1: Proposition 2 Calibration Parameter Values

Parameter	Source	Value	Comments
Goods expenditure share	BLS (2021)	$\delta = 0.8$	Similar to other quantitative spatial model calibration exercises. For instance, Redding (2016) sets $\delta = 0.75$.
Goods elasticity of substitution	Lemma 1(*ii*) via BLS (2021)	$\sigma = 5.0$	Consistent with σ selected in Allen and Arkolakis (2014), Redding (2016), and Fujita, Krugman, and Venables (1999).
Agglomeration externality	Lemma 1(*ii*) via BLS (2021)	$\alpha = 0.21$	Lower end of the range permitted by Lemma 1(*ii*).
Urban labour share	BEA (2021); ERS (2020)	$\mu_c = 0.85$	Urban/rural designation arising from county-level RUCC groupings.
Urban TFP Elasticity of In-Migration	Hornbeck and Moretti (2021)	$\beta_{\overline{L}A_c} = 2.0$	Median elasticity between Houston, San Jose, and Cincinnati identified by Hornbeck and Moretti (2021).

These elasticities imply a sensible calibration would be in the neighbourhood of two, so I choose $\beta_{\overline{L}A_c} = 2$.

Calibrated Bounds (Θ_2 and Θ_3). My choice of parameter values based upon the data, literature, and convention are summarised in Table A.1. Substituting these values into the inequality (2.11) in Proposition 2's reveals (2.11) is satisfied under these calibrated values:

$$\underbrace{2.0}_{=\beta_{\overline{L}A_c}} < \underbrace{24.45}_{=(1/(\alpha^2\eta)-1)\mu_c}$$

Substituting the calibrated values into the equations for Θ_2 and Θ_3 yields:

$$\Theta_2 = 1.16$$
$$\Theta_3 = 1.31$$

Given $\Theta_3 > \Theta_2$, the binding constraint is that $(A_c L_c^\alpha / A_r) > \Theta_3$. Thus, for Proposition 2 to hold under this parameter regime, the marginal productivity of labour ratio must be greater than 1.31, meaning a worker in the city must be no less than 31% more productive in the city than in the rural town.

Moretti (2011) finds substantial county-level manufacturing TFP heterogeneity across the U.S., reporting that the most productive county in their sample is 2.9 times more productive than the least productive county, giving weight to the possibility that the gap between urban and rural TFP may be large enough to satisfy the lower bound identified in this calibration exercise.